Mathematical Methods in Physics HW2

1. Consider the linear transformation of D on the vector space P$_3$ with representative element $x = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$. Find the matrix form of D with respect to the basis $\{1, t, t^2, t^3\}$. Now find the matrix form of the operator D^2 in two ways. First by considering the action of D^2 on x. And second by relating the matrix representation of D to the matrix representation of D^2.

Let's start by considering $Dx = \alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2$. For the 4-tuple $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ we will need

$$Dx = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 2\alpha_2 \\ 3\alpha_3 \\ 0 \end{pmatrix} \Rightarrow D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now to get D^2 we can start with $D^2 x = 2\alpha_2 + 6\alpha_3 t$ and argue

$$D^2 x = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2\alpha_2 \\ 6\alpha_3 \\ 0 \\ 0 \end{pmatrix} \Rightarrow D^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Or we can just multiply $D^2 = D * D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

2. This one is simple and straightforward, but I think the results are best absorbed if you just work out little examples of the results I gave you on determinants. Show the following:

a) A common factor in each row or column may be factored out, hence show that $detA = det \begin{pmatrix} 6a & 2b \\ 3c & d \end{pmatrix} = 6\, detB$ where $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$det \begin{pmatrix} 6a & 2b \\ 3c & d \end{pmatrix} = 3det \begin{pmatrix} 2a & 2b \\ c & d \end{pmatrix} = 6det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 6(ad - bc)$

b) If any row or column is zero then $detA = 0$. Calculate $detA$ where $A = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ and when $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. $det \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = 0 - 0 = 0$ and $det \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = 0 - 0 = 0$

c) Interchanging two rows or columns changes the sign of the determinant. Calculate $detB$ when $B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ and $detC$ when $C = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$ as well as $detD$ when $D = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ and compare to $detA$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - da = det \begin{pmatrix} b & a \\ d & c \end{pmatrix} = -det \begin{pmatrix} d & c \\ b & a \end{pmatrix} = -det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

d) If any two rows or columns are equal, then $detA = 0$. So evaluate $detA$ for $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$ and $detB$ for $B = \begin{pmatrix} a & a \\ c & c \end{pmatrix}$.

$det \begin{pmatrix} a & b \\ a & b \end{pmatrix} = ab - ab = 0$ and $det \begin{pmatrix} a & a \\ c & c \end{pmatrix} = ac - ac = 0$

e) Show explicitly that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ then $\det(AB) = \det(A)\det(B)$.

$$det\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] = det\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$
$$= (ae+bg)(cf+dh) - (af+bh)(ce+dg)$$
$$= aecf + aedh + bgcf + bgdh - afce - afdg - bhce - bhdg$$
$$= aedh + bgcf - afdg - bhce$$
$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} det\begin{pmatrix} e & f \\ g & h \end{pmatrix} = (ad - bc)(eh - fg) = adeh - adfg - bceh + bcfg$$

f) A scalar multiple of a row or column may be added to another row or column without changing the determinant. Hence show that $detA = det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = det\begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix} =$
$det\begin{pmatrix} a+3b & b \\ c+3d & d \end{pmatrix} = det\begin{pmatrix} a+2c+3b+6d & b+2d \\ c+3d & d \end{pmatrix}$.

$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$det\begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix} = (a+2c)d - (b+2d)c = ad + 2cd - bc - 2dc = ad - bc$

$det\begin{pmatrix} a+3b & b \\ c+3d & d \end{pmatrix} = (a+3b)d - b(c+3d) = ad + 3bd - bc - 3bd = ad - bc$

$det\begin{pmatrix} a+2c+3b+6d & b+2d \\ c+3d & d \end{pmatrix} = (a+2c+3b+6d)d - (b+2d)(c+3d)$

$= ad + 2cd + 3bd + 6d^2 - bc - 3bd - 2dc - 6d^2 = ad - bc$

g) If the row or column "vectors" of a matrix are linearly dependent, then $detA = 0$. So evaluate

$detA$ when $A = \begin{pmatrix} a & b \\ -3a & -3b \end{pmatrix}$.

$det\begin{pmatrix} a & b \\ -3a & -3b \end{pmatrix} = -3ab + 3ab = 0$

3. Now things get a bit harder. These are proofs, not examples.

a) Use any of the earlier results (a-e) to prove the next to last result (f).

Suppose we had a matrix $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and we wanted to add a the second row times k to

the first row, that is $M' = \begin{pmatrix} a+kd & b+ke & c+kf \\ d & e & f \\ g & h & i \end{pmatrix}$. One way to do this is to multiply the

original matrix by the following: $BM = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+kd & b+ke & c+kf \\ d & e & f \\ g & h & i \end{pmatrix}$.

Now we use the property that $det(BM) = det(B)det(M)$. We have $det(B) = 1$ so $det(BM) = det(M)$. Clearly this works for any row or column and in any dimension.

b) Use any of the earlier results (a-f) to prove the last result (g).

Suppose we start with $M = \begin{pmatrix} a & b & k(a+b) \\ d & e & k(d+e) \\ g & h & k(g+h) \end{pmatrix}$ where obviously the last column vector is

linear combination of the first two. We can now use property (f) to subtract k versions of the

first two columns from the third to obtain $M' = \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{pmatrix}$. By property (b), clearly $det M' = 0$, but through property (f) this must be the same as $det M$. Hence, if the set of column or row vectors is linearly dependant, then $det M = 0$.

4. Consider a linear transformation which takes vectors in $\mathbb{R}^3$ and projects them onto a plane defined by the axis through $x = y = z = 1$ and the origin. Even though this is a projection, an inner product will not be necessary in solving it.

a) Write down the matrix version of this linear transformation. And give an example of acting upon a vector with this operator to create the projected version of the vector. Don't use the trivial cases, i.e. a vector along the axis or a vector already in the plane. The matrix form of the projection is $P_{111} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. Note that the projection of a vector which has no components along the plane is $P_{111}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The projection of a vector which is entirely in the plane is $P_{111}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. And finally the projection of a vector along the x-axis is $P_{111}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$.

b) Does this linear transformation have an inverse? No it does not since it is not one-to-one and the determinant equals zero.

Now, imagine a change of basis which takes the axis defining the plane and aligns it with the x-axis.

c) Start by finding the operator matrix which takes vector components into their new form. We want an active rotation matrix R_a which carries the (normalized) vector $x = \alpha_i x_i$ through $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ into the vector $x' = \alpha'_i x_i$ through (1,0,0). Note that this is not a coordinate change yet since we are keeping the same basis! What happens to the rest of the vector is irrelevant.

So we want $R_a \alpha = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha'$ which of course means that $a + b + c = \sqrt{3}$, $d + e + f = 0$ and $g + h + i = 0$. This doesn't seem very useful, but if we instead consider

$$R_a^{-1}\alpha' = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \alpha,$$ then we find that $a = b = c = \frac{1}{\sqrt{3}}$. To get values for

the rest of the inverse rotation we can just use the (hopefully well known) result that $R_a^{-1} = \hat{R}_a$ (that's transpose) for rotations and hence $R_a\hat{R}_a = I$ which leads us to a set of six distinct equations given by: $d^2 + g^2 = \frac{2}{3}, e^2 + h^2 = \frac{2}{3}, f^2 + i^2 = \frac{2}{3}$ and $de + gh = -\frac{1}{3}, df + gi = -\frac{1}{3}, ef + ih = -\frac{1}{3}$.

Now we have freedom to choose how the rest of the vectors align after rotation so let's just

choose $d = \sqrt{\frac{2}{3}}$ which leads to $g = 0, e = -\frac{1}{\sqrt{6}}, f = -\frac{1}{\sqrt{6}}, h = \frac{1}{\sqrt{2}}, i = -\frac{1}{\sqrt{2}}$.

Therefore the rotation matrix which carries the axis from going through $(1,1,1)$ to the x-axis **can**

be given by $R_a = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Let's check that this works acting on the vector through $(1,1,1)$ we get

$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ which is along the x-axis, and acting on a vector in the plane

we get $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{\frac{3}{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ which is in the yz-plane.

d) Now find the transformed version of the projection operator. Now the transformed version of the projection operator is given by: $R_p^{-1}P_{111}R_p$, where R_p indicates that now we are using the passive form of the rotation. We should be careful with the result from the previous part. In that we were thinking of the rotation matrix as an active transformation which only transformed the vector components and left the basis unchanged. However, for a change of basis, we want the basis vectors rotated by R_p and the components rotated by R_p^{-1}, which means that for a transformation of the basis based on the rotation obtained in part (d), we need to identify $R_a = R_p^{-1}$ and of course $R_p^{-1} = R_a$. Then we use that an operator, under a basis change, transforms as $P' = R_p^{-1}P_{111}R_p = R_aP_{111}R_a^{-1} =$

$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

e) Verify that the transformed version of this story matches the pre-transformed version.

5. Evaluate the classical adjoint of $M = \begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}$.

$$adjM = \begin{pmatrix} (-1)^{1+1}(ej - fi) & (-1)^{2+1}(bj - ci) & (-1)^{3+1}(bf - ce) \\ (-1)^{1+2}(dj - fh) & (-1)^{2+2}(aj - ch) & (-1)^{3+2}(af - cd) \\ (-1)^{1+3}(di - eh) & (-1)^{2+3}(ai - bh) & (-1)^{3+3}(ae - bd) \end{pmatrix}$$

6. Supposing that M^{-1} in the previous question exists, calculate it.

$$M^{-1} = \frac{1}{detM}[adjM] = \frac{1}{a(ej - if) - b(dj - fh) + c(di - eh)}\begin{pmatrix} ej - fi & ci - bj & bf - ce \\ fh - dj & aj - ch & cd - af \\ di - eh & bh - ai & ae - bd \end{pmatrix}$$

7. We found in an example in class that a rotation in the xy-plane in $\mathbb{R}^3$ has three distinct eigenvalues and three distinct eigenvectors. Prove that the same is true for any rotation in $\mathbb{R}^3$.

8. Is the following matrix diagonalizable? $M = \begin{pmatrix} 3 & 0 & 0 & -1 \\ -\frac{3}{\sqrt{2}} & 2 & 0 & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix}$ Explain your reasoning.

Consider the eigenvalues:

$\det(M - \lambda I) = (3 - \lambda)(2 - \lambda)(4 - \lambda)(3 - \lambda) + 1(-2 + \lambda)(4 - \lambda)$

$= (2 - \lambda)(4 - \lambda)[(3 - \lambda)(3 - \lambda) - 1]$

$= (2 - \lambda)(4 - \lambda)[9 - 6\lambda + \lambda^2 - 1]$

$= (2 - \lambda)(4 - \lambda)[8 - 6\lambda + \lambda^2]$

$= (2 - \lambda)(4 - \lambda)(2 - \lambda)(4 - \lambda) = 0 \implies \lambda_1 = \lambda_2 = 2, \ \lambda_3 = \lambda_4 = 4 .$

Now the eigenvectors:

$Mx_{1,2} = 2x_{1,2} \implies \begin{pmatrix} 3 & 0 & 0 & -1 \\ -\frac{3}{\sqrt{2}} & 2 & 0 & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \\ 2d \end{pmatrix}$

$\implies \begin{array}{l} 3a - d = 2a \\ -\frac{3}{\sqrt{2}}a + 2b - \frac{3}{\sqrt{2}}d = 2b \\ 4c = 2c \\ -a + 3d = 2d \end{array} \implies \begin{array}{l} a = d \\ a = -d, b = anything \\ c = 0 \\ a = d \end{array} \implies \begin{array}{l} a = 0 \\ b = anything \\ c = 0 \\ d = 0 \end{array}$

$Mx_{3,4} = 4x_{3,4} \implies \begin{pmatrix} 3 & 0 & 0 & -1 \\ -\frac{3}{\sqrt{2}} & 2 & 0 & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \\ 4c \\ 4d \end{pmatrix}$

$\implies \begin{array}{l} 3a - d = 4a \\ -\frac{3}{\sqrt{2}}a + 2b - \frac{3}{\sqrt{2}}d = 4b \\ 4c = 4c \\ -a + 3d = 4d \end{array} \implies \begin{array}{l} a = -d \\ 2b = 4b \\ c = anything \\ a = -d \end{array} \implies \begin{array}{l} a = anything \\ b = 0 \\ c = anything \\ d = -a \end{array}$

The fact that the geometric multiplicity of $\lambda = 2$ is only 1, gives us three total eigenvectors which do not span the space. Therefore the linear operator M is not diagonalizable.

In fact if you try, one of the closest you will get is $M' = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.